Common Graphs

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Overview

$$x \mapsto \frac{ax+b}{cx+d}$$
$$x \mapsto a^x$$
$$x \mapsto \log_a x$$

Note that $x \mapsto x^2$ is shorthand for $f(x) = x^2$.

Möbius Transformations

Möbius transformations are a special type of rational function.

$$x \mapsto \frac{ax+b}{cx+d}$$

For a shorthand, I'll sometimes denote this function by the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For example, the following four notations denote the same rational function:

$$f(x) = \frac{3x+5}{2x-4}$$
$$x \mapsto \frac{3x+5}{2x-4}$$
$$\begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 10 \\ 4 & -8 \end{bmatrix}$$

Special Cases

If $c = 0$, the function is linear.

Note that if $ax+b$ is actually a multiple of $cx+d$, then the function is a constant. In other words, if $\frac{a}{c} = \frac{b}{d}$. This can be rearranged to $ad - bc = 0$.

We will exclude these special cases because they are ~~useless~~ trivial.

Intercepts

The y-intercept is at $(0, \frac{b}{d})$. The x-intercept is at $(0, -\frac{b}{a})$.

Limiting Behavior

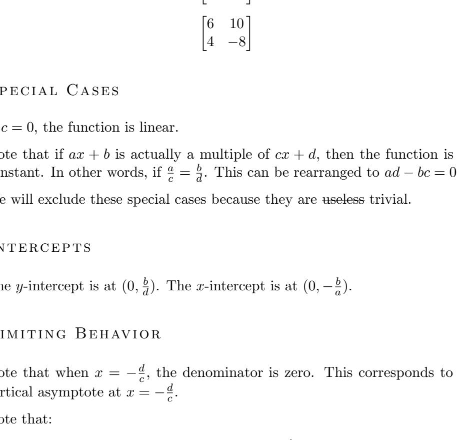
Note that when $x = -\frac{d}{c}$, the denominator is zero. This corresponds to a vertical asymptote at $x = -\frac{d}{c}$.

Note that:

$$\begin{aligned} \lim_{x\to\infty} \frac{ax+b}{cx+d} &= \lim_{x\to\infty} \frac{a + \frac{b}{x}}{c + \frac{d}{x}} \\ &= \frac{a + \lim_{x\to\infty} \frac{b}{x}}{c + \lim_{x\to\infty} \frac{d}{x}} \\ &= \frac{a}{c} \end{aligned}$$

The same result is obtained when $x \to -\infty$, so the function has one horizontal asymptote at $y = \frac{a}{c}$.

Example



- ❖ What is the equation of the horizontal asymptote?
- ❖ What is the equation of the vertical asymptote?
- ❖ What is the x-intercept?
- ❖ What is the y-intercept?

Assume that $a = 1$. From this, you should be able to deduce b, c, and d. The transformation shown above is:

$$x \mapsto \frac{x + \square}{\square x + \square}$$

One True Hyperbola

In the above example, we note that the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 1$. What would happen if we shifted the graph so these asymptotes lined up with the coordinate axes?

We start with the original transformation shown in the example.

$$f(x) = \frac{x+2}{x-2}$$

We shift the graph 2 units left and 1 unit down.

$$\begin{aligned} f(x+2) - 1 &= \frac{(x+2)+2}{(x+2)-2} - 1 \\ &= \frac{x+4}{x} - 1 \\ &= \frac{4}{x} \end{aligned}$$

It turns out that our Möbius transformation f is simply the hyperbola $x \mapsto \frac{1}{x}$, vertically scaled by 4 and shifted by $\langle 2, 1 \rangle$.

In fact, all Möbius transformations can be written in this way. We start with an arbitrary Möbius transformation.

$$f(x) = \frac{ax+b}{cx+d}$$

We note that the vertical asymptote is $x = -\frac{d}{c}$ and the horizontal asymptote is $y = \frac{a}{c}$. We shift the graph $\frac{d}{c}$ units right and $\frac{a}{c}$ units down.

$$\begin{aligned} f\left(x - \frac{d}{c}\right) - \frac{a}{c} &= \frac{a(x - \frac{d}{c}) + b}{c(x - \frac{d}{c}) + d} - \frac{a}{c} \\ &= \frac{a(cx-d)+bc}{c^2(x - \frac{d}{c}) + dc} - \frac{a}{c} \\ &= \frac{acx - ad + bc}{c^2 x} - \frac{acx}{c^2 x} \\ &= -\frac{ad-bc}{c^2}\frac{1}{x} \end{aligned}$$

Therefore, any Möbius transformation is simply the graph of $x \mapsto \frac{1}{x}$, vertically scaled by $-\frac{ad-bc}{c^2}$ and shifted by $\langle -\frac{d}{c}, \frac{a}{c} \rangle$. In other words, any such rational function is a shifted and scaled version of the well-known hyperbola $x \mapsto \frac{1}{x}$. There is one true hyperbola.

How to Graph a Möbius Transformation

1. Draw and label the axes.
2. Check if the transformation falls into a ~~useless~~ trivial case.
3. Draw the vertical asymptote at $x = -\frac{d}{c}$.
4. Draw the horizontal asymptote at $y = \frac{a}{c}$.
5. Plot the x-intercept at $(-\frac{b}{a}, 0)$.
6. Plot the y-intercept at $(0, \frac{b}{d})$.
7. Draw a scaled, translated copy of $x \mapsto \frac{1}{x}$ that has the correct asymptotes and passes through the correct intercepts.

Properties

All nontrivial Möbius transformations:

- ❖ are one-to-one
- ❖ are invertible
- ❖ are monotonic
- ❖ have one vertical and one horizontal asymptote
- ❖ are transformed copies of $x \mapsto \frac{1}{x}$

Monotonicity

The graph of the above example occupies quadrants I and III when the asymptotes are shifted to match the axes; this is equivalent to the property that the graph is everywhere downwards sloping. Some graphs of Möbius transformations, such as $x \mapsto -\frac{1}{x}$, are always upward sloping and occupy quadrants II and IV relative to the asymptotes.

How can we decide whether a transformation is upward or downwards sloping? Take the derivative, of course.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\frac{ax+b}{cx+d} &= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} \\ &= \frac{acx + ad - acx - bc}{(cx+d)^2} \\ &= \frac{ad-bc}{(cx+d)^2} \end{aligned}$$

In other words, the graph is upwards sloping precisely when $ad - bc > 0$, and vice-versa.

Function Composition

Define the transformations:

$$\begin{aligned} f(x) &:= \frac{3x+5}{2x-4} \\ g(x) &:= \frac{6x-1}{-x+5} \end{aligned}$$

What is $(g \circ f)(x) := g(f(x))$?

$$\begin{aligned} g(f(x)) &= \frac{6f(x) - 1}{-f(x) + 5} \\ &= \frac{6\frac{3x+5}{2x-4} - 1}{-\frac{3x+5}{2x-4} + 5} \\ &= \frac{6(3x+5) - (2x-4)}{-(3x+5) + 5(2x-4)} \\ &= \frac{18x + 30 - 2x + 4}{-3x - 5 + 10x - 20} \\ &= \frac{16x+34}{7x-25} \end{aligned}$$

Alternatively,

$$\frac{6x-1}{-x+5} \circ \frac{3x+5}{2x-4} = \frac{16x+34}{7x-25}$$

$$\begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 16 & 34 \\ 7 & -25 \end{bmatrix}$$

For second semester — there is a group isomorphism between the group of non-constant Möbius transformations under function composition and the projective linear group of 2×2 matrices.

For Fun

https://www.desmos.com/calculator/iuog2hqff8

Exponential Functions

$$x \mapsto a^x$$

We will (mostly) study the case when $a > 0$.

Degenerate Cases

If $a = 1$, then the function is constant. From now on, we will assume that $a \neq 1$.

Intercepts

Exponential functions where $a > 0$ are always positive, and so have no x-intercepts. All exponentials pass through the point $(0, 1)$, and so have one y-intercept.

Limiting Behavior

If $a > 1$, we have that $a^x \to \infty$ as $x \to \infty$, and $a^x \to 0$ as $x \to -\infty$. If $a < 1$, these behaviors are reversed. Therefore, the exponential function always has exactly one horizontal asymptote at $y = 0$.

Monotonicity

Suppose that $x_2 > x_1$. If $a > 0$, then $a^{x_2} > a^{x_1}$, so the graph is always increasing; if $a < 0$, the graph is always decreasing. In fact, we can show that the derivative of an exponential is proportional to itself:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0}\frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h\to 0}\frac{a^h - 1}{h} \end{aligned}$$

If we let $C := \lim_{h\to 0}\frac{a^h-1}{h}$, it follows that $\frac{\mathrm{d}}{\mathrm{d}x}a^x = Ca^x$. If we set $x = 0$ in Ca^x, it follows that C is the derivative of a^x at $x = 0$. We define the number e so that $C = 1$ when $a = e$.

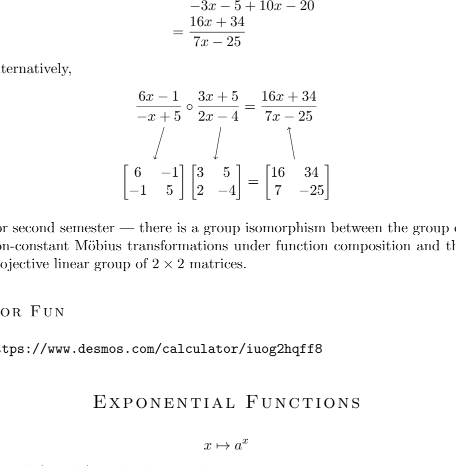
Since $a^x > 0$ for all x, it follows that its derivative must be always positive or always negative, depending on the sign of C. This implies that a^x is one-to-one and invertible.

Duality

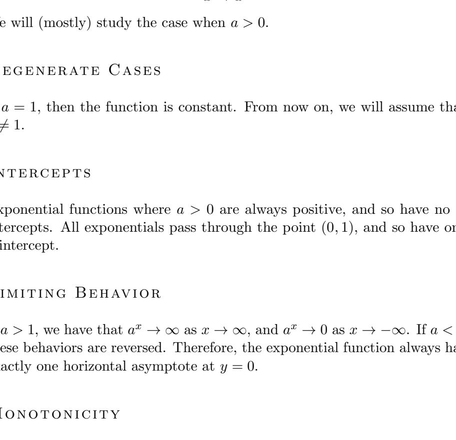
> Every law of exponents corresponds to a property of an exponential graph.

In the following examples, we will define $f_a(x) := a^x$.

Consider the exponent law $a^{m+n} = a^m a^n$. If we take $n = x$, we have that $f_a(x+m) = a^{m+x} = a^m a^x = a^m f_a(x)$. In other words, a left shift of an exponential graph is equivalent to a vertical stretch (and vice-versa).



If we take $a^{mn} = (a^m)^n$ with $n = x$, we find that $f_a(mx) = a^{mx} = (a^m)^x = f_{a^m}(x)$. A horizontal compression is equivalent to a change of base.



Similar relationships hold for all the other laws of exponents.

In particular, the relationship between horizontal scaling and change of base means that all exponential graphs are scaled versions of other exponential graphs. There is one true exponential.

Properties

All nontrivial exponentials:

- ❖ pass through $(0, 1)$
- ❖ have no x-intercepts
- ❖ are one-to-one
- ❖ are invertible
- ❖ are monotonic
- ❖ have one horizontal asymptote
- ❖ are scaled copies of each other

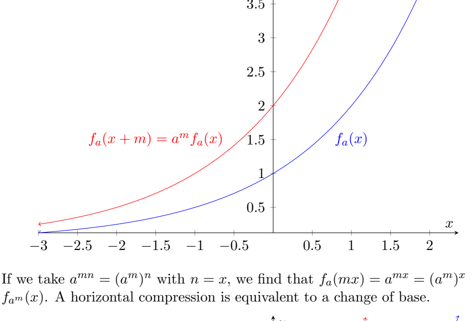
For Fun

https://www.desmos.com/calculator/ccz1puood4

Logarithmic Functions

$$x \mapsto \log_a x$$

Logarithms are the inverse of exponential functions. Therefore, most properties of exponentials (including duality) translate directly to those of logarithms. We will again consider only $a > 0$. If $a = 1$, the logarithm is undefined.



Properties

Directly translating the properties of exponentials into properties of logarithms, all nontrivial logarithms:

- ❖ pass through $(1, 0)$
- ❖ have no y-intercepts
- ❖ are one-to-one
- ❖ are invertible
- ❖ are monotonic
- ❖ have one vertical asymptote
- ❖ are scaled copies of each other

By duality (or laws of logarithms), we have that:

- ❖ vertical shifting is equivalent to horizontal stretching
- ❖ vertical scaling is equivalent to change of base

Monotonicity

Since exponentials are monotonic, logarithms are monotonic. However, we lose the property that exponentials are multiples of their own derivative. We can find the derivative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\log_a x &= \lim_{h\to 0}\frac{\log_a(x+h) - \log_a(x)}{h} \\ &= \lim_{h\to 0}\frac{\log_a(\frac{x+h}{x})}{h} \\ &= \lim_{t\to\infty}\frac{t}{x}\log_a\left(1 + \frac{1}{t}\right) && \text{(Defining } t := \tfrac{x}{h}\text{)} \\ &= \frac{1}{x}\lim_{t\to\infty} t\log_a\left(1 + \frac{1}{t}\right) \\ &= \frac{1}{x}\lim_{t\to\infty}\log_a\left(\left[1 + \frac{1}{t}\right]^t\right) \\ &= \frac{1}{x}\log_a\left(\lim_{t\to\infty}\left[1 + \frac{1}{t}\right]^t\right) && \text{(Since } \log_a x \text{ is continuous)} \end{aligned}$$

Similar to the exponential case, we define $C := \lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^t$ and obtain $\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{\log_a C}{x}$. It turns out that $C = e$.

Although the derivative has the same monotonicity property as that of exponentials, the derivative is not a multiple as its parent function. Rather than an exponential, the derivative of a logarithm is a Möbius transformation.